REM:

$$P(E) = \frac{\exp[-E^2/N]}{\sqrt{\pi N}} \Rightarrow \{E_j\}_{j=1}^{2^N} \sim \mathcal{N}(0, N/2) \quad \text{iid}$$

$$\mu_\beta(j) = \frac{\exp[-\beta E_j]}{Z} \qquad Z = \sum_{j=1}^{2^N} \exp[-\beta E_j]$$

X_N is self-averaging (concentrates) if

$$\lim_{N\to\infty} \mathbb{P}\left[\left|\frac{X_N}{N} - \frac{\mathbb{E}X_N}{N}\right| > \theta\right] = 0$$

X_N here is extensive thermodynamic potential

$\Rightarrow \mathbb{E}X_N$ provides a good description of X_N as N gets large

$$\mathcal{I} \in [N\varepsilon, N(\varepsilon+\delta)]$$

$$P[E_i \in \mathcal{I}] = \frac{1}{\sqrt{\pi N}} \int_{N\varepsilon}^{N(\varepsilon+\delta)} e^{-x^2/N}\, dx$$

$$= \sqrt{\frac{N}{\pi}} \int_\varepsilon^{\varepsilon+\delta} e^{-Nx^2} =: P_{\mathcal{I}}$$

$n(\varepsilon, \varepsilon+\delta)$ is binomial

$$\mathbb{E}\, n(\varepsilon, \varepsilon+\delta) = 2^N P_{\mathcal{I}} \quad ✓$$

$$\mathrm{Var}\; n(\varepsilon, \varepsilon+\delta) = 2^N P_{\mathcal{I}}(1-P_{\mathcal{I}}) \quad ☆$$

$$\Rightarrow \mathbb{E}\, n_\varepsilon \doteq \exp\left[N \max_{x\in[\varepsilon,\varepsilon+\delta]} (\log 2 - x^2)\right]$$

$$\mathrm{Var}\; n_\varepsilon \doteq \exp\left[N \max_{x\in[\varepsilon,\varepsilon+\delta]} (\log 2 - x^2)\right]$$

$$\Rightarrow \frac{\mathrm{Var}\, n_\varepsilon}{(\mathbb{E}n_\varepsilon)^2} \doteq \exp\left[-N \max_{x\in[\varepsilon,\varepsilon+\delta]} (\log 2 - x^2)\right]$$

Def:

$$s_a := \begin{cases} \log 2 - \varepsilon^2 & \text{if } \varepsilon < \varepsilon_* = \sqrt{\log 2} \\ -\infty & \text{otherwise} \end{cases}$$

Thm: $\lim_{N\to\infty} \frac{1}{N} \log n(\varepsilon, \varepsilon+\delta) = \sup_{x\in[\varepsilon,\varepsilon+\delta]} s(x)$

Proof: For $\varepsilon \notin [-\varepsilon_*, \varepsilon_*]$, $s(\varepsilon) < 0$

$$P[n_\varepsilon > 0] \leq \mathbb{E}\, n \doteq e^{Ns}$$

Markov — exponential decay

For $\varepsilon \in [-\varepsilon_*, \varepsilon_*]$

$$P\left[\left|\frac{n_\varepsilon}{\mathbb{E} n_\varepsilon} - 1\right| > \theta\right] \leq \frac{\operatorname{Var} n_\varepsilon}{\theta^2 \, (\mathbb{E} n_\varepsilon)^2} \doteq e^{-Ns}$$

Chebyshev — decay $\Rightarrow$ concentrates

Exercise 5.1 : $n_{out}(\delta) = \#\{j : |E_j| > N(\varepsilon_* + \delta)\}$

$$\mathbb{P}[n_{out} > 0] = 2\, \mathbb{P}[n_{[\varepsilon,\varepsilon+\infty]} > 0] \leq \mathbb{E}\, n_{[\varepsilon,\varepsilon+\infty]} \doteq e^{N \underbrace{\max_{x > \varepsilon_* + \delta} s(x)}_{\text{neg w/ gap}}}$$

$\Rightarrow$ decay w/ N

$$\Rightarrow Z_N(\beta) = \int_{-\varepsilon_*}^{\varepsilon_*} \exp\left[N\left(s(\varepsilon) - \beta\varepsilon\right)\right] d\varepsilon$$

$$\Rightarrow \phi(\beta) = \max_{\varepsilon} \; s(\varepsilon) - \beta\varepsilon$$

ϕ concentrates

$$\Rightarrow s'(\varepsilon) = \beta \Rightarrow \varepsilon = -\beta/2$$ if a sol'n exists

else $\varepsilon = -\varepsilon_* = -\sqrt{\log 2}$

$$\Rightarrow f(\beta) = -\frac{1}{\beta}\phi(\beta) = \begin{cases} -\frac{\sqrt{\log 2}}{\beta} - \frac{1}{4}\beta & \beta < \beta_c \\ -\sqrt{\log 2} & \beta > \beta_c \end{cases} \qquad \beta_c = 2\sqrt{\log 2}$$

s grows

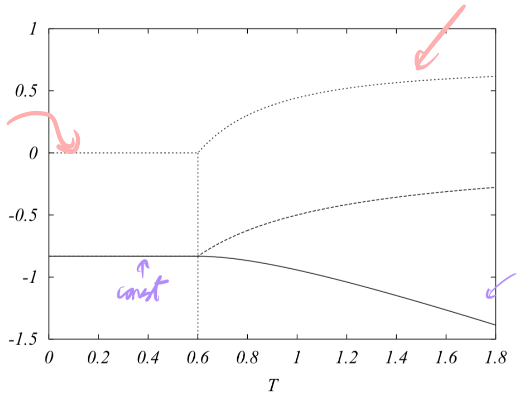


Fig. 5.3 Thermodynamics of the REM: the free-energy density (full line), the energy density (dashed line) and the entropy density (dotted line), are plotted versus the temperature $T = 1/\beta$. A phase transition takes place at $T_c = 1/(2\sqrt{\log 2}) \approx 0.6005612$.

N

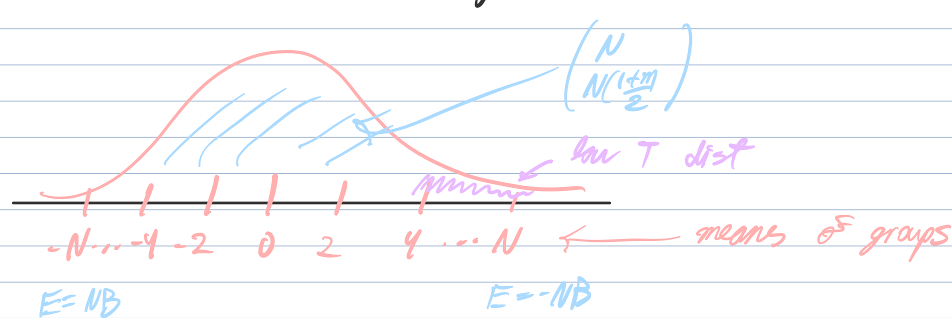
Exercise 5.2

Take the 2 configs divide into N+1 groups

each group has $M = \{-N, -N+2, \ldots, N-2, N\}$ and $\binom{N}{\frac{N+M}{2}}$ configs

$\{E_j\}$ are indep w/ var $N/2$

mean $\mathbb{E} E_j = -MB$ $\qquad m = M/N$



$$P(E_j \in \varepsilon N) = \sqrt{\frac{N}{\pi}} \int_{\varepsilon}^{\varepsilon+\delta} e^{-N(x+Bm)^2} dx \doteq \exp[-N(\varepsilon+Bm)^2]$$

$$\mathbb{E}\, n_\varepsilon = \binom{N}{mN} P_I$$

$$\doteq \exp N\left(\mathcal{H}\left(\tfrac{1+m}{2}\right) - (\varepsilon+Bm)^2\right)$$

$$\text{Var} = \binom{N}{N(\frac{1+m}{2})} P_I (1 - P_I)$$

$$\Rightarrow \int_{-1}^{1} dm \exp\left[N\left(\mathcal{H}\left(\tfrac{1+m}{2}\right) - (\varepsilon+Bm)^2\right)\right]$$

if $B=0$ this gives

$\exp[N(\log 2 - \varepsilon^2)]$

$$\Rightarrow \quad Z = \int d\epsilon \int dm \, \exp\left[N\left(\mathcal{H}\left(\tfrac{1+m}{2}\right) - (\epsilon + Bm)^2 - \beta\epsilon\right)\right]$$

Do ϵ integral: $\quad -2(\epsilon + Bm) = \beta \quad \Leftarrow @\beta = 0$

this fails for ϵ negative enough (β large enough)

$$\Rightarrow \varphi = \mathcal{H}\left(\tfrac{1+m}{2}\right) - \frac{\beta^2}{4} + \beta\left(\frac{\beta}{2} + Bm\right)$$

$$= \mathcal{H}\left(\tfrac{1+m}{2}\right) + \frac{1}{4}\beta^2 + Bm\beta$$

$2\sqrt{H\left(\tfrac{1+m}{2}\right)} = \beta_c$

the m that dominates sets β_c

or $\cancel{\mathcal{H}\left(\tfrac{1+m}{2}\right) - \mathcal{H}\left(\tfrac{1+m}{2}\right)} + \beta\left(Bm + \sqrt{\mathcal{H}\left(\tfrac{1+m}{2}\right)}\right)$ at fixed m

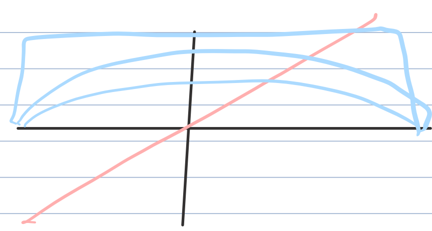
$\epsilon \in -Bm \pm \sqrt{\mathcal{H}\left(\tfrac{1+m}{2}\right)}$

Do m integral: One $m_* = \mathbb{E}M$ dominates at large N

$$\beta_c = 2\sqrt{H\left(\tfrac{1+m_c(B)}{2}\right)}$$

$$\beta < \beta_c \Rightarrow m_* = \tanh B\beta$$
$$\beta > \beta_c \Rightarrow m_* = \tanh B\beta_c$$

self consistency: $\quad m = \tanh B\sqrt{\mathcal{H}\left(\tfrac{1+m}{2}\right)}$



$\Leftarrow B = 0.1, 1, 10$

Exercise 5.3

Take $\quad P(E) \propto \exp(-C_N |E|^\delta)$

$$P(E \in [\epsilon, \epsilon+\delta]) \doteq \exp[-C_N N^\delta \epsilon^\delta]$$

$$n_\epsilon = 2^N \cdot \exp[-C_N N^\delta \epsilon^\delta]$$

$$= \exp[N(\log 2 - \hat{C} \cancel{N^{1-\delta}} \cancel{N^{\delta-1}} |\epsilon|^\delta)]$$

$\Leftarrow$ otherwise either $n_\epsilon = \delta(\epsilon)$ or n_ϵ = uniform

$$\max_\epsilon \; s(\epsilon) - \beta\epsilon$$

$$\epsilon = -\left(\frac{\beta}{\delta\hat{C}}\right)^{\frac{1}{\delta-1}}$$

$$-\frac{\log 2}{\beta}+\hat{c}\left[\left(\frac{1}{\delta\hat{c}}\right)-1\right]\left(\frac{1}{\delta\hat{c}}\right)^{\frac{1}{\delta-1}}\beta^{\frac{1}{\delta-1}}$$

Finish

5.3 Condensation of measure

$\beta>\beta_c \Rightarrow$ smaller than exp # of configs contribute

$$Y_N := \mathbb{E}[p] = \sum_{j=1}^{2^N}\mu_\beta(j)^2 = \frac{Z(2\beta)}{Z(\beta)^2}$$

as $\beta\to 0$ $\quad Y_N\to 0$

as $\beta\to\infty$ $\quad Y_N$ becomes large & fluctuates

$$\mathbb{E}_{E_j} Y = 2^N\cdot \mathbb{E}\,\mu_\beta(E_1)^2$$

$$= 2^N\,\mathbb{E}\,\frac{e^{-2\beta E_1}}{Z^2}$$

$$= 2^N\,\mathbb{E}\int dt\, t\exp\left[-2\beta E_i - t\sum_{j=1}^{2^N} e^{-\beta E_j}\right]$$

$$= 2^N\int dt\, t\,\mathbb{E}_{E_1}\exp[-2\beta E_1 - t e^{-\beta E_1}]\;\mathbb{E}_{E_{-1}}\left[e^{-tEe^{-\beta E}}\right]^{2^N-1}$$

$$a=\int P(E)\exp[-2\beta E - t e^{-\beta E}]\,dE$$

$$=\frac{1}{\sqrt{\pi N}}\int\exp\left[-\frac{E^2}{N}-2\beta E - t e^{-\beta E}\right]dE$$

make E as large and negative

take E close to E_*

$\Rightarrow t = e^{-\beta E_*}$

$e_* = \beta/2$

$\frac{\beta}{2}-\frac{2}{\beta}\log\sqrt{\pi N}$

$\Rightarrow \varepsilon_0 = -\frac{\beta_c}{2}$ at leading order

as possible

let ε_0 be estimate of ground state:

$$2^N P(-N\varepsilon_0) = 1 \Rightarrow P(w) \sim$$